

Science Article

Restricted Three-Body Problem Considering Perturbations of Oblate Primaries

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In this paper, the effect of perturbations of oblate primaries in the Circular Restricted Three-Body Problem is studied, and the equations of satellite orbital motion in the Circular Restricted Three-Body Problem are developed by employing Lagrangian mechanics. Since the equations have no closed-form solution and numerical methods must be applied, the problem can have different periodic or quasi-periodic solutions depending on the equation's initial conditions of orbital state parameters. For this purpose, an algorithm named "orbital correction algorithm" is proposed to correct the initial conditions of orbital state parameters. The limited number of periodic orbits in the study environment indicates the algorithm's need for suitable initial guesses as input. In the present paper, suitable initial guesses for orbital state parameters are selected from the third-order approximation of the Unperturbed Circular Restricted Three-Body Problem's periodic solutions, increasing the chance of obtaining desired periodic solutions. The obtained perturbed and unperturbed periodic orbits are compared in order to understand the effect of perturbations. Adding the perturbations brings the study environment closer to the real environment and helps understand satellites' natural motion.

Keywords: Three-body problem, Perturbations of oblate primaries, Periodic solutions, Lagrangian points

Introduction

Today, space agencies are increasingly using multi-body dynamical structures in their missions. These missions are performed with aims such as astronomical observations (J.P. Gardner et al., 2006; J. Krist, 2007; M. Hechler and J. Cobos, 2003), building space habitats, improving the pointing accuracy of space telescopes, and docking with space stations (M. Machula and G. Sandhoo, 2005). The Circular Restricted Three-Body Problem can be an appropriate approximation of

the mentioned multi-body structures in which a spacecraft is afloat in the gravitational field of two primary planets, and its presence does not affect their motion (H.D. Curtis, 2013). The reason why the primaries' motions are not affected by the spacecraft is that the spacecraft's mass is considered infinitesimal compared to that of the primaries.

Valuable research has been done on the Circular Restricted Three-Body Problem. Kunistyn studied the stability of satellites in the vicinity of the Libration points (A.L. Kunitsyn, 2013). Kikuchi and Tsuda performed the three-axis stabilization of

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satellites in the Circular Restricted Three-Body Problem (S. Kikuchi et al., 2019). Lega and Guzzo studied the energy manifolds in the Circular Restricted Three-Body Problem (E. Lega and M. Guzzo, 2016). Bakhtiari and Abbas Ali studied satellite flight formation in the Elliptical Restricted Three-Body Problem (M. Bakhtiari et al., 2017).

The satellite equations of motion in the Circular Restricted Three-Body Problem have no closed-form solutions (H.D. Curtis, 2013;) and therefore require numerical solving methods. In numerical methods, depending on the initial conditions of the state parameters, the problem can have countless solutions. Therefore, a numerous number of orbits can be found in this problem. Orbits with periodic and quasi-periodic structures have many applications, including keeping satellites in trajectory and data transmission continuity.

Halo and Lyapunov orbits are two well-known types of periodic orbits in the Circular Restricted Three-Body Problem (X. Hou et al., 2018; B. Wong et al., 2008). Robert Farquhar first used the term 'Halo orbits' in his doctoral dissertation (R.W. Farquhar, 1968). Farquhar and Kamel then produced analytical solutions to find quasi-periodic orbits in the vicinity of the L_2 point in the Earth-Moon system (R.W. Farquhar and A.A. Kamel, 1973) using the method of Lindstedt-Poincaré (A. Casal and M. Freedman, 1980). Following these studies, Breakwell and Brown developed a numerical method for finding stable periodic orbits in the vicinity of the L_2 point of the Earth-Moon system (J. V Breakwell and J. V Brown, 1979). Using these studies, Howell developed a comprehensive numerical method for finding Halo orbits in the Lagrangian points of the Circular Restricted Three-Body Problem in his doctoral dissertation (K.C. Howell, 1984). Bringing the study environment closer to the real environment will lead to a more accurate simulation of the above orbits. Adding the spatial perturbations to the study environment helps achieve this goal (Y.-J. Qian et al., 2019). Singh calculated the orbital analysis of the Circular Restricted Three-Body Problem regime considering the oblate Earth in the Earth-Moon system (J. Singh, 2015). Srivastava and Kumar regularized the orbits of the Circular Restricted Three-Body Problem system with radiation pressure and the oblate Earth in the Earth-Sun system (V.K. Srivastava et al., 2017), using

Lagrangian mechanics in order to solve the equations of orbital motion. Markellos and Papadakis studied the nonlinear stability of satellites in the vicinity of the Lagrangian points of the Earth-Moon system considering the perturbations of oblate Earth (V. V Markellos et al., 1996). Singh completed the research by accounting for a spacecraft with variable masses (J. Singh, 2015). Zhang et al. studied the effect of oblateness perturbations on spacecraft hovering in low Earth orbit (L. Zhang and P. Ge, 2021).

This paper aims to investigate the effect of perturbations of both oblate primaries on the periodic satellite orbits in the Circular Restricted Three-Body Problem for the Earth-Moon system. Accounting for the perturbations of oblate Earth and oblate Moon will bring the study environment closer to the real environment and lead to a better simulation and understanding of these orbits. This paper will refer to the Perturbed Circular Restricted Three-Body Problem and the Unperturbed Circular Restricted Three-Body Problem as P-CRTBP and U-CRTBP, respectively. Comparing the periodic orbits of these two environments will better understand the effect of primaries' oblateness perturbations.

The geometry of the Perturbed Circular Restricted Three-Body Problem

Assume two oblate primaries with masses m_1 and m_2 ($m_1 > m_2$), which only move under the influence of each other's mutual gravitation. We add a spacecraft with the mass of m to this system and want to express its equations of motion in the system. The mass of the spacecraft is negligible compared to that of the primaries. In celestial mechanics, this problem is referred to as the Perturbed Circular Three-Body Problem because the spacecraft's motion does not affect the motion of the primaries (point in case, the mass of spacecraft versus the mass of the planets of the solar system). The motion of the small spacecraft m in this system is the most important part of the Circular Restricted Three-Body Problem. When two primaries move in a particular circular orbit, the coordinates constant to their motions can be defined. The center of these coordinates is located on the barycenter of primaries. The coordinates rotate at constant angular velocity Ω equal to the mean motion of the primaries. These coordinates, known as a rotating frame, are expressed by unit vectors $\mathbf{r}(\hat{x}, \hat{y}, \hat{z})$ in such a way that \hat{z} is

perpendicular to the primaries' planes of motion and the primaries always remain on the \hat{x} -axis. The inertial frame $\mathbf{I}(\hat{X}, \hat{Y}, \hat{Z})$ is defined so that at the $t = 0$ initial time, it is aligned to the rotating frame and \hat{Z} is always aligned with the direction of rotating frame \hat{z} unit vector and is perpendicular to the primaries' plane of motion. Now, for simplification of equations, it is better to use a non-dimensional form so that the universal gravitation constant G , angular velocity Ω and the distance between the two primaries are unity. We'll also assume the orbital period of the primaries as 2π and the mass parameter μ as the ratio between m_2 and the total mass of the system $\mu = \frac{m_2}{m_1 + m_2}$.

Mass parameter μ also defines the location of the primaries in a way that primary m_1 is located at $(-\mu, 0, 0)$ and primary m_2 is located at $(1 - \mu, 0, 0)$. The assumed data used in non-dimensionalizing the equations can be summarized in the following relations:

$$\begin{aligned} l' &= Ll \\ v' &= \frac{L}{T} v \\ t' &= \frac{T}{2\pi} t \end{aligned} \quad (1)$$

Where the primed parameters are dimensional, the unprimed parameters are non-dimensional, and the distance between the primaries and their orbital period of motion are L and T , respectively.

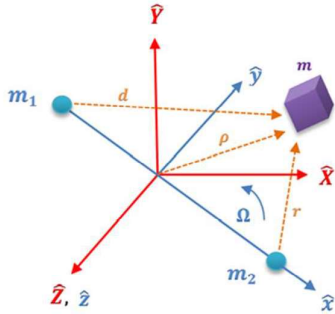


Figure 1. The geometry of the P-CRTBP. The inertial frame and rotating frame are represented by red and blue, respectively.

Equations of spacecraft orbital motion in the P-CRTBP with perturbations of both oblate primaries

In this paper, Lagrangian mechanics is the main method used for obtaining the equations of orbital motion in the P-CRTBP. In this mechanics, a body's equations of motion are extractable by obtaining kinetic energy and potential field

affecting the object. The Lagrangian function L is commonly written as:

$$L = V - K \quad (2)$$

Where K and V are kinetic energy and potential field of the system, then the following equations are used to obtain the equations of motion:

$$\frac{\partial L}{\partial f(x,y,z)} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{f}(x,y,z)} \right) = 0 \quad (3)$$

The potential field at P-CRTBP considering the perturbations of both oblate primaries

The potential gravitational field per unit mass of a body at the distance ρ from the center of the oblate body with the mean radius of R_m can be written as (V.K. Srivastava, J. Kumar, B.S. Kushvah, 2017):

$$V = -\frac{G}{\rho} \left[1 - \sum_{l=1}^{\infty} J_{2l} \left(\frac{R_m}{\rho} \right)^{2l} P_{2l}(\cos \theta) \right] \quad (4)$$

Where J_{2l} is the second zonal harmonics, θ is the angle from the satellite to the center of the oblate primary, and P_{2l} is the Legendre polynomials.

According to research, since the second zonal harmonic J_2 is much larger than other zonal harmonics, the present paper only consider this harmonic and ignores the effect of other harmonics.

Accounting for the second zonal harmonic J_2 , the potential field of the spacecraft located at distance r from the oblate primary is obtained by placing $k = 2$ in Equation (4) and, after simplification, is rewritten as follows:

$$V = -Gm \left(\frac{1}{r} + \frac{A_2}{2r^3} \right) \quad (5)$$

Where A_2 is the primary's oblateness coefficient and is equal to $J_2 R_e^2$, and R_e is the primary's equatorial radius (V.K. Srivastava, J. Kumar, B.S. Kushvah, 2017).

Considering both primaries as oblate bodies, the potential field of the P-CRTBP is obtained by non-dimensionalizing Equation (5), which is the sum of potential fields of both oblate primaries:

$$V = \left(-\mu \left(\frac{1}{r} + \frac{A_2^{(2)}}{r^3} \right) \right) + \left(-(1 - \mu) \left(\frac{1}{d} + \frac{A_2^{(1)}}{d^3} \right) \right) \quad (6)$$

Where d and r are the non-dimensional distance of the spacecraft from m_1 and m_2 , respectively:

$$\begin{aligned} d &= \sqrt{(x + \mu)^2 + y^2 + z^2} \\ r &= \sqrt{(x - 1 + \mu)^2 + y^2 + z^2} \end{aligned} \quad (7)$$

The kinetic energy in P-CRTBP considering the perturbations of both oblate primaries

The kinetic energy of a spacecraft in a rotating frame which rotates about z -axis with uniform angular velocity n , is expressed as (V.K. Srivastava, J. Kumar, B.S. Kushvah, 2017):

$$K = \frac{1}{2}(v_x^2 + v_y^2 + v_z^2) + n(xy\dot{y} - \dot{x}y) + \frac{1}{2}n^2(x^2 + y^2) \quad (8)$$

Where v_x , v_y , and v_z are velocity components of the spacecraft in rotating frame. Mean motion n is obtained from (J.A. Arredondo, J. Guo, C. Stoica, C. Tamayo, 2012):

$$n = \sqrt{1 + 3(J_2^{(1)} + J_2^{(2)})} \quad (10)$$

Where $J_2^{(1)}$ and $J_2^{(2)}$ are zonal harmonics of m_1 and m_2 , respectively.

Formulating the equations of spacecraft orbital motion in the P-CRTBP with perturbations of oblate primaries

As mentioned in section 4, the method used in this paper to obtain the equations of spacecraft orbital motion $f_x(\mathbf{x}_{orb})$ is Lagrangian mechanics. The orbital state vector $\mathbf{x}_{orb} = [x, y, z, v_x, v_y, v_z]$ includes the spacecraft's center of mass position and velocity components expressed in the rotating frame. The equations of spacecraft orbital motion in the P-CRTBP are obtained by placing relations (6) and (8) in function (2) and using Equation (3):

$$\dot{\mathbf{x}}(\mathbf{x}_{orb}) = \begin{bmatrix} \ddot{x} = n^2x + 2nv_y + (\mu - 1) \left[\frac{x+\mu}{d^3} + 3A_2^{(1)} \frac{x+\mu}{d^5} \right] - \mu \left[\frac{x-1+\mu}{r^3} + 3A_2^{(2)} \frac{x-1+\mu}{r^5} \right] \\ \ddot{y} = n^2y - 2nv_x + (\mu - 1) \left[\frac{y}{d^3} + 3A_2^{(1)} \frac{y}{d^5} \right] - \mu \left[\frac{y}{r^3} + 3A_2^{(2)} \frac{y}{r^5} \right] \\ \ddot{z} = (\mu - 1) \left[\frac{z}{d^3} + 3A_2^{(1)} \frac{z}{d^5} \right] - \mu \left[\frac{z}{r^3} + 3A_2^{(2)} \frac{z}{r^5} \right] \end{bmatrix} \quad (10)$$

Note that by setting the oblateness coefficient to zero in Equation (10), the equations of satellite orbital motion in the unperturbed case are obtained. The equation system used in (10) which, in fact, expresses the spacecraft orbital motion in the P-CRTBP, has no closed-form solution and therefore requires numerical solving methods. Numerical methods require initial conditions to begin the solution process. The problem can have different answers based on the initial condition, including periodic or quasi-periodic. Obtaining periodic orbits in the P-CRTBP is one of the most important aims of this research, seeing that it helps

us to understand the effect of perturbations on the orbits. Since there is a limited number of periodic orbits in the U-CRTBP, it is expected that the number of periodic orbits in the P-CRTBP would be limited as well. Also, since based on the initial conditions assumed for the orbital state vector \mathbf{x}_{orb} , solving the equation system does not lead to periodic answers in many cases, we need to find suitable initial conditions to reach periodic answers. Kathleen Howell obtained some of these periodic orbits, known as Halo orbits, in the U-CRTBP (K.C. Howell, 1984).

In this paper, the orbital correction algorithm is proposed in order to achieve the suitable initial conditions of periodic Halo and Lyapunov orbits in the P-CRTBP. This algorithm uses State Transition Matrix (STM) of the equations of orbital motion to correct the initial guesses assumed for the initial condition of \mathbf{x}_{orb} orbital state vector. The mentioned initial guesses, which are derived from the third-order approximation of equations of orbital motion in the U-CRTBP, are available in (K.C. Howell, 1984; D. Guzzetti and K.C. Howell, 2014)

P-CRTBP orbital correction algorithm

Depending on the initial conditions of the orbital state vector, several periodic orbits are obtained in this problem. It is expected that some of the periodic orbits found in the U-CRTBP would be found in the P-CRTBP as well. These orbits, known as Halo and Lyapunov families, have special characteristics such as being symmetrical and perpendicular to the orbital plane of the primaries. In this paper, we will use the above characteristics to obtain the orbital motion algorithm of the P-CRTBP in order to achieve the suitable initial conditions of the Halo and Lyapunov families.

P-CRTBP Halo orbit family

Assume that the orbit of the oblate primaries is located on xz plane. In general, the initial vector of orbital motion state is written as $\bar{\mathbf{X}}_0 = [x_0, y_0, z_0, v_{x_0}, v_{y_0}, v_{z_0}]$. The initial guess vector is perpendicular to xz plane if it is written as $\bar{\mathbf{X}}_0 = [x_0, 0, z_0, 0, v_{y_0}, 0]^T$ with an orbital period of T . If another crossing from xz plane is defined as $\bar{\mathbf{X}}(\frac{T}{2}) = [x, 0, z, 0, v_y, 0]$, then the orbit is periodic, symmetrical, and perpendicular to the plane of motion of the primaries throughout its trajectory.

Orbits with these characteristics are known as Halo orbits.

Considering the initial condition, numerical integration of equations of motion (10) is used until y changes its sign. Next, if $v_x = 0$ and $v_z = 0$, then periodic Halo orbit is obtained. Otherwise, if $v_x \neq 0$ and $v_z \neq 0$, then we assume that the initial guess correction vector must be written as $[\delta x_0, 0, \delta z_0, 0, \delta v_{y_0}, 0]$. If $y(\frac{T}{2}) = 0$, then δv_{y_0} and δv_{x_0} must be modified to correct the initial guess in order to achieve the desired initial condition. The State Transition Matrix of the orbital equations ($STM_o = \Phi_o$) at $(\frac{T}{2})$ can represent the relation between the initial guess vector at the initial time and at half-period time. This matrix is generally defined as follows, where J_o is the Jacobian of Equation (10) in the state space, and $I_{6 \times 6}$ is the identity matrix.

$$\Phi_o(t, 0) = J_{6 \times 6} \Phi_o \quad (11)$$

$$\Phi_o(0, 0) = I_{6 \times 6}$$

Orbital correction algorithm is calculated from (K.C. Howell, 1984):

$$\delta \bar{X} = \Phi_o\left(\frac{T}{2}, 0\right) \delta \bar{X}_0 + \frac{\partial \bar{X}}{\partial t} \delta\left(\frac{T}{2}\right) \quad (12)$$

Note that:

$$\delta y = 0 = \Phi_{o_{21}} x_0 + \Phi_{o_{23}} z_0 + \Phi_{o_{25}} v_{y_0} + v_y \delta\left(\frac{T}{2}\right) \quad (13)$$

Now, if $x_0 > z_0$, it is better to leave z_0 fixed and only change v_{y_0} and x_0 :

$$\begin{pmatrix} \delta v_x \\ \delta v_z \end{pmatrix} = \begin{bmatrix} \Phi_{o_{43}} & \Phi_{o_{45}} \\ \Phi_{o_{63}} & \Phi_{o_{65}} \end{bmatrix} - \frac{1}{v_y} \begin{pmatrix} \dot{v}_x \\ \dot{v}_z \end{pmatrix} \begin{pmatrix} \Phi_{o_{21}} & \Phi_{o_{25}} \end{pmatrix} \begin{pmatrix} \delta x_0 \\ \delta v_{y_0} \end{pmatrix} \quad (14)$$

And if $z_0 > x_0$, it is better to leave x_0 fixed and only change v_{y_0} and z_0 :

$$\begin{pmatrix} \delta v_x \\ \delta v_z \end{pmatrix} = \begin{bmatrix} \Phi_{o_{41}} & \Phi_{o_{45}} \\ \Phi_{o_{61}} & \Phi_{o_{65}} \end{bmatrix} - \frac{1}{v_y} \begin{pmatrix} \dot{v}_x \\ \dot{v}_z \end{pmatrix} \begin{pmatrix} \Phi_{o_{23}} & \Phi_{o_{25}} \end{pmatrix} \begin{pmatrix} \delta z_0 \\ \delta v_{y_0} \end{pmatrix} \quad (15)$$

P-CRTBP Lyapunov orbit family

In Lyapunov orbits, which are two-dimensional trajectories, the initial guess vector is written as $\bar{X}_0 = [x_0, 0, 0, 0, v_{y_0}, 0]^T$, which is perpendicular to xz plane at the initial time, T is the orbital period. If another crossing from xz plane is defined as $\bar{X}(\frac{T}{2}) = [x, 0, 0, 0, v_y, 0]$, then the orbit is periodic, symmetrical, and perpendicular to the plane of

motion of the primaries throughout its trajectory. The process of extracting the orbital correction algorithm is the same as Halo orbits, but with $v_z = \delta z_0 = 0$, due to their two-dimensionality.

Results

In this paper, the Earth and Moon are considered as oblate primaries. Table 1 shows the constant values of this system.

Table 1. Constant values of oblate Earth-Moon system

$\mu_{Earth-Moon}$	0.0121505856
$J_2^{(earth)}$	1.0826×10^{-3}
$J_2^{(moon)}$	2.0323×10^{-4}
$D_{Earth-Moon}(km)$	384400
$R_{eEarth}(km)$	6378.1
$R_{eMoon}(km)$	1738.1

As stated before, extracting the periodic orbits in the P-CRTBP is one of this research aims. The orbital correction algorithm is used to achieve the initial conditions of P-CRTBP Halo and Lyapunov orbits. This algorithm uses the results of the third-order approximation of equations of motion for the U-CRTBP available in (K.C. Howell, 1984; D. Guzzetti and K.C. Howell, 2014) as suitable inputs. Table 2 shows some of this algorithm's outputs, including the initial conditions of the Halo and Lyapunov periodic orbits in the P-CRTBP in the vicinity of the Libration points.

In this paper, the a th periodic orbits of the Halo and Lyapunov families in the vicinity of the e th Lagrangian points are represented by $H(e:a)$ and $L(e:a)$, respectively, and non-dimensional units are represented by ($ndim$).

Table 2. The corrected initial conditions for $\bar{x}_0^c(L)$ and $\bar{x}_0^c(H)$ periodic orbits with orbital period T of the Earth-Moon system obtained by the proposed orbital correction algorithm with the initial guess vectors of $\bar{x}_0^g(L)$ and $\bar{x}_0^g(H)$ orbit families.

	x_0 (ndim)	y_0 (ndim)	z_0 (ndim)	v_{x_0} (ndim)	v_{y_0} (ndim)	v_{z_0} (ndim)	T (ndim)
$\bar{x}_0^g(L(1:1))$	0.7689	0	0	0	0.423	0	3.95
$\bar{x}_0^c(L(1:1))$	0.7815732	0	0	0	0.4432	0	
$\bar{x}_0^g(L(1:2))$	0.7816	0	0	0	0.47	0	4.4
$\bar{x}_0^c(L(1:2))$	0.7688476	0	0	0	0.4813	0	
$\bar{x}_0^g(L(2:1))$	1.22	0	0	0	-0.41	0	4.31
$\bar{x}_0^c(L(2:1))$	1.2199794	0	0	0	-0.4275	0	
$\bar{x}_0^g(L(2:2))$	1.2248	0	0	0	-0.44	0	4.46
$\bar{x}_0^c(L(2:2))$	1.2248305	0	0	0	-0.4419	0	
$\bar{x}_0^g(L(3:1))$	-1.6068	0	0	0	1.1	0	6.23
$\bar{x}_0^c(L(3:1))$	-1.6068057	0	0	0	1.1159	0	
$\bar{x}_0^g(L(3:2))$	-1.6392	0	0	0	1.1669	0	6.25
$\bar{x}_0^c(L(3:2))$	-1.6392143	0	0	0	1.1745	0	
$\bar{x}_0^g(H(1:1))$	0.8234	0	0.0224	0	0.1343	0	3
$\bar{x}_0^c(H(1:1))$	0.81764301	0	0.02002	0	0.0987	0	
$\bar{x}_0^g(H(1:2))$	0.85	0	0.044	0	0.15	0	3.23
$\bar{x}_0^c(H(1:2))$	0.8432776	0	0.0433	0	0.1589	0	
$\bar{x}_0^g(H(2:1))$	1.17	0	0.011	0	-0.1432	0	5
$\bar{x}_0^c(H(2:1))$	1.1807092	0	0.01389	0	-0.15696	0	
$\bar{x}_0^g(H(2:2))$	1.01	0	0.17	0	-0.1	0	5.287
$\bar{x}_0^c(H(2:2))$	1.02748	0	0.18562	0	-0.1148	0	
$\bar{x}_0^g(H(3:1))$	-0.71	0	0.5	0	-0.19	0	6.25
$\bar{x}_0^c(H(3:1))$	-0.7916	0	0.6160	0	-0.2129	0	
$\bar{x}_0^g(H(3:2))$	-0.5	0	0.791	0	-0.39	0	6.3
$\bar{x}_0^c(H(3:2))$	-0.5798	0	0.8160	0	-0.4241	0	

Some periodic orbits are shown in Figures (8) and (9) based on the conditions obtained in Table 2. The similarity between these orbits and the Halo and Lyapunov orbit families in the U-CRTBP available in literature (A. Celletti, G. Pucacco, D. Stella, 2015; E. Canalias and J.J. Masdemont, 2006) can be considered as a validation for the obtained responses.

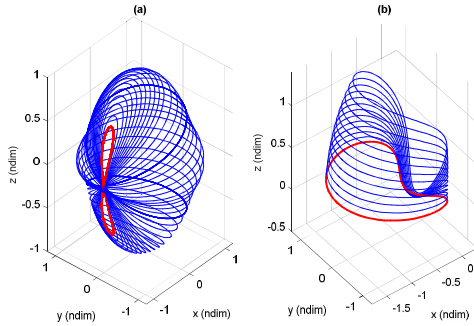


Figure 2. The P-CRTBP L_3 (a) Vertical Halo orbit family (b) Axial Halo orbits family. The first orbit of each family is represented by red.

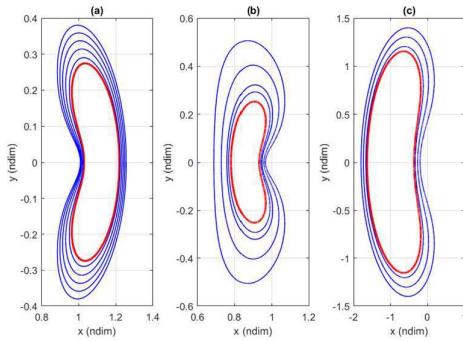


Figure 3. The P-CRTBP Lyapunov orbit family in the vicinity of (a) Libration Point L_1 , (b) Libration Point L_2 , (c) Libration Point L_3 . The first orbit of each family is represented by red.

this paper, Lyapunov periodic orbits are used to show the effect of the previously mentioned perturbations. Table 3 shows the absolute difference between the initial conditions of orbit state parameters of the P-CRTBP and the U-CRTBP for the Lyapunov orbit family to show the importance of initial conditions in drawing the periodic tables and demonstrate the effect of perturbations of both oblate primaries.

Table 3. The absolute difference Δ between the initial conditions of Lyapunov orbits in the U-CRTBP and the P-CRTBP with perturbations of both oblate primaries in the vicinity of the collinear Libration points.

orbit	Δ (ndim)
$L(1:1)$	$\Delta x_0^* = 3.7603 \times 10^{-7}$ $\Delta v_{y_0}^* = 0$
$L(1:2)$	$\Delta x_0^* = 3.5774 \times 10^{-7}$ $\Delta v_{y_0}^* = 0$
$L(1:3)$	$\Delta x_0^* = 3.766 \times 10^{-7}$ $\Delta v_{y_0}^* = 0$
$L(2:1)$	$\Delta x_0^* = 1.0139 \times 10^{-6}$ $\Delta v_{y_0}^* = 0$
$L(2:2)$	$\Delta x_0^* = 1.0297 \times 10^{-6}$ $\Delta v_{y_0}^* = 0$
$L(2:3)$	$\Delta x_0^* = 0$ $\Delta v_{y_0}^* = 1.4812 \times 10^{-6}$
$L(3:1)$	$\Delta x_0^* = 9.3765 \times 10^{-7}$ $\Delta v_{y_0}^* = 0$
$L(3:2)$	$\Delta x_0^* = 7.3958 \times 10^{-7}$ $\Delta v_{y_0}^* = 0$
$L(3:3)$	$\Delta x_0^* = 4.5108 \times 10^{-7}$ $\Delta v_{y_0}^* = 0$

According to Table 3, the absolute difference Δ between the U-CRTBP and the P-CRTBP models may seem insignificant, but the information presented in Figure 4 refutes this claim. In Figure (4), to show the importance of initial conditions and the effect of perturbations, the initial conditions for equations of the orbital motion of the unperturbed model are used as the initial conditions for the equations of the orbital motion of the perturbed model. Figure 4 (a and b) show the perturbed models in the vicinity of the Libration point L_1 and L_2 with the conditions previously mentioned diverging after the second orbital period. Figure 4 (c) shows that moving away from the primaries delays divergence to sixteen periods. This is evident in Lyapunov orbits in the vicinity of the Libration point L_3 , which is more distant from the primaries than other Libration points. Based on Table 3 and Figure 4, it is noteworthy that the accuracy for drawing periodic orbits according to the initial conditions of the orbital state is up to seven digits in the non-dimensional form.

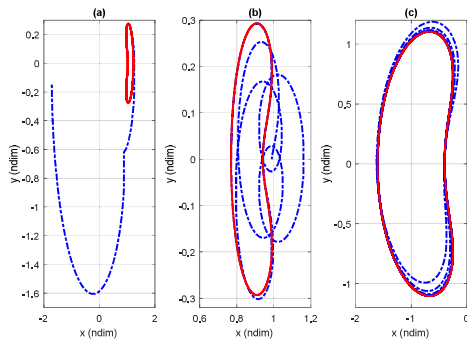


Figure 4. Initial conditions of unperturbed cases were used as the perturbed model's initial conditions for drawing Earth-Moon system Lyapunov orbits. Lyapunov orbits in the vicinity of the Libration point L_1 (a), Lyapunov orbits in the vicinity of the Libration point L_2 (b), and Lyapunov orbits in the vicinity of the Libration point L_3 (c).

Applying the effect of perturbations of both oblate primaries is the main reason for the changes in the initial conditions of Lyapunov orbits in the P-CRTBP, which, as shown in Figure 10, has a significant effect on the temporal behavior of these reference trajectories.

Obtaining mean orbital error in the P-CRTBP relative to U-CRTBP can illustrate the effect of perturbations of both oblate primaries during one orbital period. This error was obtained by averaging the state parameter errors at 100 points in both models concerning identical initial guesses of both models (orbital correction algorithm inputs).

Table 4. Mean error Δ_{Eav} of orbital state parameters in the unperturbed model relative to the perturbed model. This error is obtained by averaging the state parameter errors at 100 points of the orbit concerning the identical initial guesses of both models.

Orbit	$\Delta_{Eav}x$ (km)	$\Delta_{Eav}y$ (km)	$\Delta_{Eav}v_x$ (km/h)	$\Delta_{Eav}v_y$ (km/h)	t (ndim)
$L(1:1)$	0.1582	0.1742	0.0096	0.0141	3.95
$L(1:2)$	0.2018	0.2291	0.0132	0.0173	4.4
$L(1:3)$	0.2374	0.2582	0.0159	0.0202	4.6
$L(1:4)$	0.3600	0.4269	0.0276	0.0308	5.31
$L(1:5)$	0.5852	0.8470	0.0442	0.0617	6.1
$L(2:1)$	0.3369	0.5962	0.0379	0.0442	4.31
$L(2:2)$	0.3117	0.5170	0.0326	0.037	4.46
$L(2:3)$	0.2823	0.3414	0.0256	0.0221	4.69
$L(2:4)$	0.5792	0.4639	0.0538	0.0435	5.12
$L(2:5)$	1.2848	2.1180	0.1813	0.1471	5.315
$L(2:6)$	4.0045	6.1486	1.1655	0.3515	6.47
$L(3:1)$	0.0139	0.018	0.0019	0.0004	6.23
$L(3:2)$	0.03	0.046	0.0021	0.0022	6.25
$L(3:3)$	0.0605	0.0788	0.0047	0.0045	6.288

Table 4 shows that the mean error in Lyapunov orbits in the vicinity of L_1 and L_3 is directly related to the orbital period. It means that, in the orbit families of these points, the mean error is increased with increasing the orbital period. This behavior is also true in the first three orbits of the Lyapunov orbit family in the vicinity of the, but after the fourth period, the opposite happens, meaning that the mean error is decreased with increasing the orbital period. This indicates the chaotic behavior of orbits in the vicinity of L_2 Libration point.

Conclusion

The main aim of this paper was to study the effect of the perturbations of both oblate primaries on periodic orbits in the Circular Restricted Three-Body Problem. The equations of orbital motion in the P-CRTBP were elicited by employing Lagrangian mechanics. Due to the lack of closed-

form solutions, we were required to use a numerical method. Numerical methods are highly dependent on initial conditions of state parameters and can have periodic or quasi-periodic answers depending on the different initial conditions. In this paper, an orbital correction algorithm was proposed to correct the initial conditions of orbital state parameters which led to obtaining the initial conditions of periodic answers. It was shown that the accuracy of the dependency of periodic solutions on initial conditions is up to seven digits. Next, periodic orbits in the P-CRTBP and U-CRTBP were compared in order to show the effect of perturbations of oblate primaries. Using the initial conditions of U-CRTBP periodic orbits in the P-CRTBP environment indicated that the existing orbits diverge from their periodic form, showing perturbations' effect on the P-CRTBP. Obtaining the mean error based on the initial guesses of both perturbed and unperturbed models illustrates the effect of perturbations during one orbital period as well. Applying the perturbations to the Circular Restricted Three-Body Problem model can bring the study environment to the real environment, which results in a more accurate simulation of the motion of satellites. This is considered a step forward in solving such problems.

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